Homework 4

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1. A set S of propositional statements is *independent* if for any $A \in S$, there is a valuation that makes all the formulas in $S \setminus \{A\}$ true and makes A false. One also says that A is *not* implied logically by the rest of the statements in S. (So, by definition, the empty set $\emptyset$ is independent, and $S = \{A\}$ is independent iff A is *not* a tautology.)

   Which of the sets

   (a) $\{p \Rightarrow q,\ q \Rightarrow r,\ r \Rightarrow q\}$

   (b) $\{p \Rightarrow q,\ q \Rightarrow r,\ p \Rightarrow r\}$

   (c) $\{p \Rightarrow r,\ r \Rightarrow q,\ q \Rightarrow p,\ r \Rightarrow (q \Rightarrow p)\}$

   are independent and which are not? (Please explain.)

2. Let G be a graph with set of vertices V. A *coloring* of G with k colors ($k = 1, 2, \dots$) is a function c that assigns to each vertex in V one of the "colors" $1, 2, \dots, k$, in such a way that if $x, y \in V$ are *adjacent* (i.e., connected by an arc of G), then $c(x) \neq c(y)$. Describe a way to assign to each G a propositional statement P_G such that G is k-colorable iff P_G is satisfiable. Explain why your statement works, and illustrate with a few examples.

3. Using resolution, show that $p \wedge q \wedge r$ is implied by the following set of formulas:

   $$\{p \Rightarrow q,\ q \Rightarrow r,\ r \Rightarrow p,\ p \vee q \vee r\}.$$

   (Recall that this means that any valuation that makes all the formulas in the set true also must make the formula $p \wedge q \wedge r$ true.)

4. Using resolution, show that

   $$(\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg r) \vee (q \wedge r) \vee p$$

   is a tautology.

5. Describe an algorithm that given two whole numbers n, m, returns the number n^m. Write $\|n\|$ for the number of digits of n and $\|m\|$ for the number of digits of m. Express in terms of $\|n\|$ and $\|m\|$ the number of steps that your algorithm requires.