Solutions

Homework #10

6.2 B.8 Conjecture: If the remainder when b is divided by a is 1, then the remainder when b^2 is divided by a is also 1.

This is equivalent to the following: If $b = aq_1 + 1$ for some integer q_1, then $b^2 = aq_2 + 1$ for some integer q_2.

Suppose $b = aq + 1$ for some integer q. Then $b^2 = (aq + 1)^2 = a^2q^2 + 2aq + 1 = a(aq^2 + 2q) + 1$. $aq^2 + 2q$ is an integer because it is a sum and product of integers, so the conclusion holds.

<u>Notes</u>:

$\frac{b}{a}$ does <u>not</u> mean "a divides b". "a divides b" is written in shorthand as $a \mid b$. For integers a and b, $a \neq 0$, $\frac{b}{a}$ is a rational number while $a \mid b$ is a statement.

The conjecture is <u>not</u> equivalent to "if $b = aq + 1$ for some integer q, then $b^2 = aq + 1$"; that statement is equivalent to "if $b = aq+1$ for some integer q, then $b^2 = b$", which is obviously false. As mentioned in section 3.6, you cannot use the same letter to represent two potentially different unknowns.

6.2 B.22 (No solution for B23- these are similar enough.)

(a) $1 = 7 - 2(3)$

(b) Since $17 = 7(2) + 3$, $3 = 17 - 7(2)$. Then

$$1 = 7 - 2(3) = 7 - 2(17 - 7(2)) = 7 - 17(2) + 7(4) = 7(5) - 17(2).$$

(c) From (b), one solution is $x_0 = -2, y_0 = 5$. All the solutions to $17x + 7y = 1$ are given by $x = -2 + 7t, y = 5 - 17t$.

To see that these are solutions, notice

$$\begin{aligned}17x + 7y &= 17(-2 + 7t) + 7(5 - 17t) \\ &= 17(-2) + 17(7t) + 7(5) + 7(-17t) \\ &= 17(-2) + 7(5) + 17(7t) + 7(-17t) \\ &= 17(-2) + 7(5) = 1 \qquad \text{using the result in part (b)}\end{aligned}$$

This answer is somewhat incomplete because it does not show that <u>all</u> solutions are given by $x = -2 + 7t, y = 5 - 17t$; however I consider this answer to be sufficient for this homework.

For those wondering how to show that all solutions are given by the above equations. . .

Let (x_0, y_0) be the solution found in part (b), $x_0 = -2$ and $y_0 = 5$, and let (x, y) be any other solution. Then

$$\begin{aligned}&17x + 7y = 17x_0 + 7y_0 \\ \implies\ & 17(x - x_0) = -7(y - y_0) \qquad (1) \\ \implies\ & -\frac{17}{7} = \frac{y - y_0}{x - x_0}\end{aligned}$$

The equality $17(x - x_0) = -7(y - y_0)$ from above implies $7 \mid 17(x - x_0)$ and $17 \mid -7(y - y_0)$.

By Euclid's Lemma (Thereom 20 in the text), since $\gcd(17, 7) = 1$, we have $7 \mid (x-x_0)$. Similarly, since $\gcd(-7, 17) = 1$, we have $17 \mid (y - y_0)$. Therefore there exist integers q_1, q_2 so that

$x - x_0 = 7q_1$ and $y - y_0 = 17q_2$.

Then $17(x - x_0) = 17(7q_1) = -7(y - y_0) = -7(17q_2) \implies q_1 = -q_2$. Thus $x - x_0 = 7q_1$ and $y - y_0 = 17q_2 = -17q_1$,

or $x = x_0 + 7q_1 = -2 + 7q_1$ and $y = y_0 - 17q_1 = 5 - 17q_1$.

6.2 B.24 Find $\gcd(10, 23)$. (No solutions for B25-B27- these are similar enough.)

$$23 = 10(2) + 3 \qquad \gcd(23, 10) = \gcd(10, 3)$$

$$10 = 3(3) + 1 \qquad \gcd(10, 3) = \gcd(3, 1)$$

$$3 = 1(3) + 0 \qquad \gcd(3, 1) = \gcd(1, 0)$$

then transitively $\gcd(23, 10) = \gcd(1, 0) = 1$.

Rearranging the second equation we get 3=23-10(2), so

$$1 = 10 - 3(3) = 10 - 3(23 - 10(2)) = 10 - 23(3) + 10(6)$$

$$= 10(7) - 23(3)$$

so a solution to $1 = 10x + 23y$ is $x = 7, y = -3$.

Notes:

Dr. Caicedo wants you to mention the gcd equalities; i.e. $\gcd(23, 10) = \gcd(10, 3)$, etc. This property is the key to the Euclidean Algorithm after all, so an answer without this is considered incomplete.

On the other hand, many of you received comments from me about the stopping case, $3 = 1(3)+0$: Dr. Caicedo does not require you to state this for full credit. I left comments before asking him his thoughts about this, but did not take off any points.

3.6 B.6 Conjecture 6 is false. Prove it.
Conjecture 6: If $S \cap T \neq \emptyset$ and $S \cap R \neq \emptyset$, then $T \cap R \neq \emptyset$.

Disprove by counterexample:

Let $S = \{1, 2\}$, $T = \{1\}$ and $R = \{2\}$. Then $S \cap T = \{1\} \neq \emptyset$ and $S \cap R = \{2\} \neq \emptyset$, but $T \cap R = \emptyset$.

Notes:

On pg. 209 of the text, section 3.6, it states: For existence proofs, in addition to exhibiting a candidate, you must verify the candidate has all the properties it is required to have.

When a problem just says "give a counterexample", you do not necessarily need to demonstrate it is a counterexample. If you are asked to prove that a conjecture is false, as in this problem, you need to demonstrate that your choice is in fact a counterexample.

3.6 B.31
Prove by induction: $(1 + n)^{3n} - 1$ is divisible by n.

Line 11 is not allowed, because we have not shown that $a - c \neq 0$ (line 11 is obtained from line 10 by dividing by $a - c$, and divison by zero is not allowed). In fact, line 2 $a = c$ implies $a - c = 0$, so the flaw in this case is doubly bad.

All other lines in the proof are valid steps. Note that line 12 does follow logically if it is assumed that line 11 is true, so this step is still okay.